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Halpern-type iterations for strongly relatively nonexpansive mappings in Banach spaces

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ABSTRACT

In this paper, we note that the main convergence theorem in Zhang et al. (2011) [21] is incorrect and we prove a correction. We also modify Halpern's iteration for finding a fixed point of a strongly relatively nonexpansive mapping in a Banach space. Consequently, two strong convergence theorems for a relatively nonexpansive mapping and for a mapping of firmly nonexpansive type are deduced. Using the concept of duality theorems, we obtain analogue results for strongly generalized nonexpansive mappings and for mappings of firmly generalized nonexpansive type. In addition, we study two strong convergence theorems concerning two types of resolvents of a maximal monotone operator in a Banach space.

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1. Introduction

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping of a closed and convex subset of a Banach space E (that is, $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$). In 1953, Mann [1] introduced the following iterative method: a sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n = 1, 2, 3, \ldots, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. It is known that under appropriate conditions the sequence $\{x_n\}$ converges only *weakly* to a fixed point of T. However, even in a Hilbert space, Mann iteration may fail to converge strongly; for example, see [2].

Several attempts to construct the iteration method guaranteeing the strong convergence have been made. For example, Halpern [3] proposed the following so-called Halpern iteration: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n, \quad n = 1, 2, 3, \ldots, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$ or $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Another approach was proposed by Bauschke and Combettes [4]. More precisely, their algorithm is defined by

$$\begin{cases} x_1 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \ldots, \end{cases}$$

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where $\limsup_{n\to\infty} \alpha_n < 1$ and P_K denotes the metric projection from a Hilbert space H onto a closed and convex subset K of H. It should be noted here that the iteration above works only in Hilbert space setting. To extend this iteration to a Banach space, a *relatively nonexpansive* mapping [5–7] was introduced. Before we give its definition, we recall some notations. The strong and weak convergences of a sequence $\{x_n\}$ in a Banach space E to an element $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let E be a smooth Banach space and let E^* be the dual of E. Denote by $\langle \cdot, \cdot \rangle$ the pairing between E and E^*. Let J be the normalized duality mapping from E to E^*. Alber [8] considered the following functional $\varphi : E \times E \to [0, \infty)$ defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$. Using this functional, Matsushita and Takahashi [5–7,9] studied and investigated the following mappings in Banach spaces. Suppose that C is a subset of a smooth Banach space E. A mapping $T : C \to E$ is *relatively nonexpansive* if the following properties are satisfied:

(R1) $F(T) \neq \varnothing$, where $F(T)$ denotes the fixed points set of T;
(R2) $\varphi(p, Tx) \le \varphi(p, x)$ for all $p \in F(T)$ and $x \in C$;
(R3) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0, it follows that $p \in F(T)$.

In a Hilbert space H, the duality mapping J is the identity mapping and $\varphi(x, y) = \|x-y\|^2$ for all $x, y \in H$. Hence, if $T : C \to H$ is a nonexpansive mapping of a nonempty, closed and convex subset C of H, then it is relatively nonexpansive.

There are many methods for approximating fixed points of relatively nonexpansive mappings (see, *e.g.*, [5–7,10–21]). In 2004, Matsushita and Takahashi [5] studied the Mann-type iteration for a relatively nonexpansive mapping defined by

$$\begin{cases} x_1 \in C \quad \text{is arbitrary,} \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \quad n = 1, 2, 3, \ldots, \end{cases}$$

where $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, the interior of $F(T)$ is nonempty and Π_C denotes the generalized projection from E onto C. Moreover, they proposed the following analogue of the Bauschke and Combettes algorithm:

$$\begin{cases} x_1 \in C \quad \text{is arbitrary,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \varphi(z, y_n) \le \varphi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_1 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \ldots, \end{cases}$$

where $\limsup_{n\to\infty} \alpha_n < 1$.

Recently, Zhang et al. [21] modify Halpern's iteration for finding fixed point of relatively nonexpansive mappings in the following result.

Theorem 1.1 (*Cf. [21, Theorem 4.1]*)**.** *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : C \to C$ be a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)JTx_n), \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. If the interior of $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to a fixed point of T.

Careful reading of the proof of Theorem 1.1, leads to the fact that the inequality (4.5) is not correct. Indeed, the assumptions, for each $u \in F(T)$,

$$\varphi(u, x_{n+1}) \le \alpha_n \varphi(u, x_1) + (1-\alpha_n)\varphi(u, Tx_n),$$

$\lim_{n\to\infty} \alpha_n = 0$ and $\varphi(u, Tx_n) \le \varphi(u, x_n)$ are *not enough* to guarantee that

$$\varphi(u, x_{n+1}) \le \varphi(u, x_n).$$

Consequently, the inequalities (4.10)–(4.15) are also not correct. Moreover, we know that the interior of the singleton fixed point set of T is empty and there are many relatively nonexpansive mappings whose fixed point sets are singleton.

We say that a relatively nonexpansive mapping $T : C \to E$ is *strongly relatively nonexpansive* [9,22] if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(p, x_n) - \varphi(p, Tx_n) \to 0$ for some $p \in F(T)$ it follows that $\varphi(Tx_n, x_n) \to 0$. Note that the notion of a strongly nonexpansive mapping with respect to the norm was first introduced and studied in [23] (see also [24]).

Example 1.2 (*Cf. [25,26]*)**.** Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $T : C \to E$ be a relatively nonexpansive mapping. Suppose that there exists $\kappa > 0$ such that

$$\varphi(p, Tx) + \kappa\varphi(Tx, x) \le \varphi(p, x)$$

for all $p \in F(T)$ and $x \in C$. Then T is a strongly nonexpansive mapping.

Many authors studied weak and strong convergence theorems of strongly relatively nonexpansive mappings (see, for instance, [10,12,19,25–30] and the references therein).

Another well-known family of mappings is the class of firmly nonexpansive mappings, where a mapping $T : C \to E$ is called *firmly nonexpansive type* [27] if

$$\varphi(Tx, Ty) + \varphi(Ty, Tx) + \varphi(Tx, x) + \varphi(Ty, y) \leq \varphi(Tx, y) + \varphi(Ty, x)$$

for all $x, y \in C$. See [19,27–31] for more information on firmly nonexpansive type mappings. It is easy to see that if T is firmly nonexpansive type with $I - T$ is demi-closed at zero, then it is strongly relatively nonexpansive. Furthermore, there is a mapping which is strongly relatively nonexpansive but is not firmly nonexpansive type as the following example shows.

Example 1.3. Let E be a smooth, strictly convex, and reflexive Banach space and let $T : E \to E$ be a mapping defined by

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ \left(\frac{2}{3} \sin \frac{1}{\|x\|}\right) x & \text{if } x \neq 0. \end{cases}$$

Then $F(T) = \{0\}$. We observe that

$$\begin{aligned} \varphi(Tx, x) &= \left(\frac{4}{9} \sin^2 \frac{1}{\|x\|}\right) \|x\|^2 - 2\left(\frac{2}{3} \sin^2 \frac{1}{\|x\|}\right) \langle x, Jx\rangle + \|x\|^2 \\ &= \left(1 - \frac{2}{3} \sin \frac{1}{\|x\|}\right)^2 \|x\|^2 \\ &\leq \left(1 + \frac{2}{3}\right)^2 \|x\|^2 = \frac{25}{9} \|x\|^2. \end{aligned}$$

Then

$$\begin{aligned} \varphi(0, Tx) &= \left(\frac{4}{9} \sin^2 \frac{1}{\|x\|}\right) \|x\|^2 \leq \frac{4}{9} \|x\|^2 \\ &= \|x\|^2 - \frac{5}{9} \|x\|^2 \\ &\leq \varphi(0, x) - \frac{1}{5} \varphi(Tx, x) \end{aligned}$$

for all $x \in E$. This implies that T is relatively nonexpansive and

$$\frac{1}{5} \varphi(Tx, x) \leq \varphi(0, x) - \varphi(0, Tx)$$

for all $x \in E$. It follows that $\varphi(Tx_n, x_n) \to 0$ whenever $\{x_n\}$ is a bounded sequence such that $\varphi(0, x_n) - \varphi(0, Tx_n) \to 0$. That is, T is strongly relatively nonexpansive. Let $x_0 \in S_E$ be fixed. Put $x = \frac{2}{\pi} x_0$ and $y = \frac{2}{3\pi} x_0$. Then $Tx = \frac{4}{3\pi} x_0$ and $Ty = -\frac{4}{9\pi} x_0$. It follows that

$$\varphi(Tx, Ty) = \frac{16}{9\pi^2} - 2\left\langle \frac{4}{3\pi} x_0, J\left(-\frac{4}{9\pi} x_0\right)\right\rangle + \frac{16}{81\pi^2} = \frac{256}{81\pi^2} = \varphi(Ty, Tx).$$

Consequently,

$$\begin{aligned} \varphi(Tx, Ty) + \varphi(Ty, Tx) + \varphi(Tx, x) + \varphi(Ty, y) &= 2\frac{256}{81\pi^2} + \frac{4}{9\pi^2} + \frac{100}{81\pi^2} \\ &= \frac{4}{9\pi^2} + \frac{612}{81\pi^2} \\ &> \frac{4}{9\pi^2} + \frac{484}{81\pi^2} = \varphi(Tx, y) + \varphi(Ty, x). \end{aligned}$$

Hence, T is not firmly nonexpansive type.

The purpose of this paper is to prove for a class of strongly relatively nonexpansive mappings that only Conditions (C1) and (C2) are sufficient for the strong convergence theorem of Halpern's iterations to a fixed point of T without the assumption of the nonempty interior of the fixed point set of T. Consequently, a strong convergence theorem for a relatively nonexpansive mapping is deduced and a correction for [21, Theorem 4.1] is presented. Using a concept of duality theorems (see, for instance, [32,33]), we obtain an analogue result for a strongly generalized nonexpansive mapping. Moreover, two corresponding strong convergence theorems for a firmly nonexpansive type mapping [27] and a firmly generalized nonexpansive type mapping [34] are deduced. Finally, we discuss two strong convergence theorems concerning two types of resolvents of a maximal monotone operator in a Banach space.

2. Preliminaries

We present several definitions and preliminaries which are needed in this paper. We say that a Banach space E is *strictly convex* if the following implication holds for any $x, y \in E$:

$$\|x\| = \|y\| = 1 \quad \text{and} \quad x \neq y \text{ imply } \left\| \frac{x+y}{2} \right\| < 1.$$

We say that E is *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{and} \quad \|x - y\| \geq \varepsilon \text{ imply } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

It is known that if E is a uniformly convex Banach space, then E is reflexive and strictly convex.

Let S_E denote the unit sphere of E, that is, $S_E := \{x \in E : \|x\| = 1\}$. The norm $\|\cdot\|$ of E is said to be *Gâteaux differentiable* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S_E$. In this case, E is said to be *smooth*. The norm of E is said to be *uniformly Gâteaux differentiable* (resp. *Fréchet differentiable*) if for each $y \in S_E$ (resp. for each $x \in S_E$) the limit (2.1) is attained uniformly for any $x \in S_E$ (resp. uniformly for any $y \in S_E$). The norm of E is said to be *uniformly Fréchet differentiable* (and E is called *uniformly smooth*) if the limit (2.1) is attained uniformly for any $x, y \in S_E$. This is well-known that

(1) if E is reflexive, then E is smooth if and only if E^* is strictly convex;
(2) E is uniformly smooth if and only if E^* is uniformly convex.

The value of $x^* \in E^*$ at $x \in E$ is denoted by $\langle x, x^* \rangle$. The *duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

We also know the following properties (see, *e.g.*, [35] for details):

(a) $J(x) \neq \varnothing$ for each $x \in E$.
(b) If E is smooth, then J is single valued.
(c) If E is strictly convex, then $J(x) \cap J(y) = \varnothing$ for all $x \neq y$.
(d) If E has a uniformly Gâteaux differentiable norm, then J is uniformly norm-to-weak* continuous on each bounded subset of E.
(e) If E has a Fréchet differentiable norm, then J is norm-to-norm continuous.
(f) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
(g) If E is a Hilbert space, then J is the identity operator.

Let E be a smooth Banach space. The function $\varphi : E \times E \to \mathbb{R}$ (see [8]) is defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2.$$

It is obvious from the definition of the function φ that

$$(\|x\| - \|y\|)^2 \leq \varphi(x, y) \leq (\|x\| + \|y\|)^2$$

and

$$\varphi\left(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\right) \leq \lambda\varphi(x, y) + (1-\lambda)\varphi(x, z) \tag{2.2}$$

for all $\lambda \in [0, 1]$ and $x, y, z \in E$. It is also easy to check that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a smooth Banach space E, then $x_n - y_n \to 0$ implies that $\varphi(x_n, y_n) \to 0$.

Lemma 2.1 (*Cf. [36, Proposition 2]*). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\varphi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.*

Remark 2.2. For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E, we have

$$\varphi(x_n, y_n) \to 0 \iff x_n - y_n \to 0 \iff Jx_n - Jy_n \to 0.$$

Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E. It is known that [8,36] for any $x \in E$ there exists a unique point $\widehat{x} \in C$ such that

$$\varphi(\widehat{x}, x) = \min_{y \in C} \varphi(y, x).$$

Following Alber [8], we denote such an element $\widehat{x}$ by $\Pi_C x$. The mapping Π_C is called the *generalized projection* from E onto C. It is easy to see that, in a Hilbert space, the mapping Π_C coincides with the metric projection P_C.

Lemma 2.3 (*Cf. [36, Propositions 4 and 5]*)**.** *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E, $x \in E$ and $\widehat{x} \in C$. Then*

(a) $\widehat{x} = \Pi_C x$ *if and only if* $\langle y - \widehat{x}, Jx - J\widehat{x}\rangle \leq 0$ *for all* $y \in C$;
(b) $\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \leq \varphi(y, x)$ *for all* $y \in C$.

Remark 2.4. The generalized projection mapping Π_C above is relatively nonexpansive and $F(\Pi_C) = C$.

Let E be a reflexive, strictly convex and smooth Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^* = J^{-1}$. We will use the following mapping $V : E \times E^* \to \mathbb{R}$ studied in [8]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.3}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \varphi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.5 (*Cf. [8] and [37, Lemma 3.2]*)**.** *Let E be a reflexive, strictly convex and smooth Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 (*Cf. [38, Lemma 2.1]*)**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n$$

for all $n \in \mathbb{N}$, where the sequences $\{\gamma_n\}$ in $(0, 1)$ and $\{\delta_n\}$ in $\mathbb{R}$ satisfy the following conditions: $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ *and* $\limsup_{n\to\infty} \delta_n \leq 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 (*Cf. [39, Lemma 3.1]*)**.** *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.8 (*Cf. [40, Lemma 1]*)**.** *Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n, \quad n = 1, 2, 3, \ldots.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3. Strongly relatively nonexpansive mappings

In this section, we use Halpern's idea [3] for finding fixed point of strongly relatively nonexpansive mappings in a uniformly convex and smooth Banach space.

A mapping $T : C \to E$ is said to be *relatively quasi-nonexpansive* [15] if it satisfies only (R1) and (R2). In a Hilbert space H, the duality mapping J is the identity mapping and $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Hence, if $T : C \to H$ is relatively quasi-nonexpansive, then it is quasi-nonexpansive, that is, $\|p - Tx\| \leq \|p - x\|$ for all $p \in F(T)$ and $x \in C$. In the sequel, we shall need the following lemmas.

Lemma 3.1 (*Cf. [15, Lemma 2.5]*)**.** *Let C be a nonempty, closed and convex subset of a strictly convex and smooth Banach space E and let $T : C \to E$ be a relatively quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 3.2. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E, $T : C \to E$ be a relatively nonexpansive mapping, $x \in E$ and $\widehat{x} = \Pi_{F(T)}x$. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences such that $\varphi(Tx_n, x_n) \to 0$ and $\varphi(Tx_n, y_n) \to 0$. Then*

$$\limsup_{n\to\infty}\langle y_n - \widehat{x}, Jx - J\widehat{x}\rangle \leq 0.$$

Proof. From the uniform convexity of E and Lemma 2.1,

$$Tx_n - x_n \to 0 \quad \text{and} \quad y_n - x_n \to 0.$$

From property (R3) of the mapping T, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup y \in F(T)$ and

$$\limsup_{n\to\infty}\langle y_n - \widehat{x}, Jx - J\widehat{x}\rangle = \limsup_{n\to\infty}\langle x_n - \widehat{x}, Jx - J\widehat{x}\rangle = \lim_{i\to\infty}\langle x_{n_i} - \widehat{x}, Jx - J\widehat{x}\rangle.$$

From Lemma 2.3(a), we immediately obtain that

$$\limsup_{n\to\infty}\langle y_n - \widehat{x}, Jx - J\widehat{x}\rangle = \langle y - \widehat{x}, Jx - J\widehat{x}\rangle \le 0. \quad \square$$

Using the technique in [16,39], we obtain the following theorem.

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E and let $T : C \to E$ be a strongly relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by $u \in E$, $x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)JTx_n), \quad n = 1, 2, 3, \ldots, \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty}\alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$.

Proof. Let

$$y_n \equiv J^{-1}(\alpha_n Ju + (1-\alpha_n)JTx_n).$$

Then $x_{n+1} \equiv \Pi_C y_n$. Since $F(T)$ is nonempty, closed and convex, we put

$$\widehat{u} = \Pi_{F(T)}u.$$

We first show that $\{x_n\}$ is bounded. From Remark 2.4 and (2.2), we have

$$\begin{aligned}\varphi(\widehat{u}, x_{n+1}) &\le \varphi(\widehat{u}, y_n)\\ &\le \alpha_n\varphi(\widehat{u}, u) + (1-\alpha_n)\varphi(\widehat{u}, Tx_n)\\ &\le \alpha_n\varphi(\widehat{u}, u) + (1-\alpha_n)\varphi(\widehat{u}, x_n)\\ &\le \max\{\varphi(\widehat{u}, u), \varphi(\widehat{u}, x_n)\}.\end{aligned}$$

By induction, we have

$$\varphi(\widehat{u}, x_{n+1}) \le \max\{\varphi(\widehat{u}, u), \varphi(\widehat{u}, x_1)\}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and so is the sequence $\{Tx_n\}$. From Condition (C1) and (2.2), we obtain

$$\varphi(Tx_n, y_n) \le \alpha_n\varphi(Tx_n, u) + (1-\alpha_n)\varphi(Tx_n, Tx_n) = \alpha_n\varphi(Tx_n, u) \to 0. \tag{3.2}$$

From Remark 2.4, Lemma 2.5 and (2.2), we have

$$\begin{aligned}\varphi(\widehat{u}, x_{n+1}) &\le \varphi(\widehat{u}, y_n) = V(\widehat{u}, Jy_n)\\ &\le V(\widehat{u}, Jy_n - \alpha_n(Ju - J\widehat{u})) - 2\langle y_n - \widehat{u}, -\alpha_n(Ju - J\widehat{u})\rangle\\ &= V(\widehat{u}, \alpha_n J\widehat{u} + (1-\alpha_n)JTx_n) + 2\alpha_n\langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle\\ &= \varphi(\widehat{u}, J^{-1}(\alpha_n J\widehat{u} + (1-\alpha_n)JTx_n)) + 2\alpha_n\langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle\\ &\le \alpha_n\varphi(\widehat{u}, \widehat{u}) + (1-\alpha_n)\varphi(\widehat{u}, Tx_n) + 2\alpha_n\langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle\\ &\le (1-\alpha_n)\varphi(\widehat{u}, x_n) + 2\alpha_n\langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle,\end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$.

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(\widehat{u}, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\varphi(\widehat{u}, x_n)\}$ is then convergent. Then

$$\varphi(\widehat{u}, x_n) - \varphi(\widehat{u}, x_{n+1}) \to 0. \tag{3.4}$$

Notice that

$$\varphi(\widehat{u}, x_{n+1}) \le \alpha_n\varphi(\widehat{u}, u) + (1-\alpha_n)\varphi(\widehat{u}, Tx_n).$$

It follows from (3.4) and Condition (C1) that

$$\begin{aligned}\varphi(\widehat{u}, x_n) - \varphi(\widehat{u}, Tx_n) &= \varphi(\widehat{u}, x_n) - \varphi(\widehat{u}, x_{n+1}) + \varphi(\widehat{u}, x_{n+1}) - \varphi(\widehat{u}, Tx_n)\\ &\le \varphi(\widehat{u}, x_n) - \varphi(\widehat{u}, x_{n+1}) + \alpha_n(\varphi(\widehat{u}, u) - \varphi(\widehat{u}, Tx_n)) \to 0.\end{aligned}$$

Since T is strongly relatively nonexpansive,

$$\varphi(Tx_n, x_n) \to 0.$$

It follows from (3.2) and Lemma 3.2 that

$$\limsup_{n\to\infty} \langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle \le 0.$$

Hence the conclusion follows from Lemmas 2.6 and 2.1, and (3.3).

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(\widehat{u}, x_{n_i}) < \varphi(\widehat{u}, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.7, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\varphi(\widehat{u}, x_{m_k}) \le \varphi(\widehat{u}, x_{m_k+1}) \quad \text{and} \quad \varphi(\widehat{u}, x_k) \le \varphi(\widehat{u}, x_{m_k+1})$$

for all $k \in \mathbb{N}$. This together with Condition (C1) gives

$$\begin{aligned}\varphi(\widehat{u}, x_{m_k}) - \varphi(\widehat{u}, Tx_{m_k}) &= \varphi(\widehat{u}, x_{m_k}) - \varphi(\widehat{u}, x_{m_k+1}) + \varphi(\widehat{u}, x_{m_k+1}) - \varphi(\widehat{u}, Tx_{m_k}) \\ &\le \alpha_{m_k}\left(\varphi(\widehat{u}, u) - \varphi(\widehat{u}, Tx_{m_k})\right) \to 0.\end{aligned}$$

This implies that

$$\varphi(Tx_{m_k}, x_{m_k}) \to 0.$$

It now follows from (3.2) and Lemma 3.2 that

$$\limsup_{k\to\infty} \langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle \le 0. \tag{3.5}$$

From (3.3), we have

$$\varphi(\widehat{u}, x_{m_k+1}) \le (1-\alpha_{m_k})\varphi(\widehat{u}, x_{m_k}) + 2\alpha_{m_k}\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle. \tag{3.6}$$

Since $\varphi(\widehat{u}, x_{m_k}) \le \varphi(\widehat{u}, x_{m_k+1})$, we have

$$\begin{aligned}\alpha_{m_k}\varphi(\widehat{u}, x_{m_k}) &\le \varphi(\widehat{u}, x_{m_k}) - \varphi(\widehat{u}, x_{m_k+1}) + 2\alpha_{m_k}\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle \\ &\le 2\alpha_{m_k}\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle.\end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we get

$$\varphi(\widehat{u}, x_{m_k}) \le 2\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle.$$

It follows from (3.5) that $\varphi(\widehat{u}, x_{m_k}) \to 0$. This together with (3.6) gives

$$\varphi(\widehat{u}, x_{m_k+1}) \to 0.$$

But $\varphi(\widehat{u}, x_k) \le \varphi(\widehat{u}, x_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $x_k \to \widehat{u}$.

This implies that $x_n \to \widehat{u}$ and the proof is complete. □

Remark 3.4. The result [41, Corollary 8] is a special case of our result.

Lemma 3.5 (*Cf. [12, Lemmas 3.1 and 3.2]*). *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to E$ be a relatively nonexpansive mapping. Let U be the mapping defined by*

$$U = J^{-1}(\lambda J + (1-\lambda)JT),$$

where $\lambda \in (0, 1)$, then $U : C \to E$ is strongly relatively nonexpansive and $F(U) = F(T)$.

Applying Theorem 3.3 and Lemma 3.5, we have the following result.

Theorem 3.6 (*Cf. [16, Corollary 5]*). *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : C \to E$ be a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by $u \in E$, $x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)(\lambda Jx_n + (1-\lambda)JTx_n))$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) and (C2), and $\lambda \in (0, 1)$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$.

Remark 3.7. In Theorems 3.3 and 3.6, the condition of the nonempty interior of fixed point set of T is not needed.

We next prove a strong convergence theorem of our iteration in the presence of another condition and give the revised version of Theorem 1.1.

Theorem 3.8. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : C \to E$ be a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by (3.1), where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$. If the interior of $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to z, where $z = \lim_{n\to\infty} \Pi_{F(T)}x_n$.*

Proof. We show only that $\{Jx_n\}$ is a Cauchy sequence. To this end, we revise inequalities (4.5) and (4.15) in the proof of [21, Theorem 4.1] as follows: for each $w \in F(T)$,

$$\varphi(w, x_{n+1}) \le \varphi(w, x_n) + \alpha_n\varphi(w, x_1),$$

and there exist $p \in F(T)$, $h \in E$ with $\|h\| \le 1$ and $r > 0$ such that $p + rh \in F(T)$ and

$$\|Jx_m - Jx_n\| \le \frac{1}{2r}(\varphi(p, x_m) - \varphi(p, x_n)) + \frac{\varphi(p + rh, x_1)}{r} \sum_{i=m}^{n-1} \alpha_i,$$

for each $m < n$, respectively. Since $\sum_{n=1}^{\infty} \alpha_n < \infty$ from Lemma 2.8, we have $\lim_{n\to\infty} \varphi(w, x_n)$ exists and so $\{Jx_n\}$ is a Cauchy sequence. The rest of the proof is similar to the proof of [21, Theorem 4.1], so it is left for the reader to verify. □

From Theorem 3.3, we apply the result for finding fixed point of a firmly nonexpansive type mapping. Since firmly nonexpansive type mappings in a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable are strongly relatively nonexpansive [27, Theorem 5.2], we have the following results.

Theorem 3.9. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let $T : C \to E$ be a firmly nonexpansive type mapping such that $F(T)$ is nonempty. Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) and (C2). Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $\Pi_{F(T)}u$.*

Theorem 3.10. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to E$ be a firmly nonexpansive type mapping such that $F(T)$ is nonempty. Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$. If the interior of $F(T)$ is nonempty, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to z, where $z = \lim_{n\to\infty} \Pi_{F(T)}x_n$.*

4. Strongly generalized nonexpansive mappings

Let C be a subset of a smooth Banach space E. In 2007, Ibaraki and Takahashi [42] introduced the following mapping. A mapping $T : C \to E$ is *generalized nonexpansive* if the following properties are satisfied:

(G1) $F(T) \neq \varnothing$;
(G2) $\varphi(Tx, p) \le \varphi(x, p)$ for all $x \in C$ and $p \in F(T)$.

A generalized nonexpansive mapping $T : C \to E$ is *strongly generalized nonexpansive* [16] if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(x_n, p) - \varphi(Tx_n, p) \to 0$ for some $p \in F(T)$ it follows that $\varphi(x_n, Tx_n) \to 0$. A mapping $T : C \to E$ satisfies *property* (G3) if whenever $\{x_n\}$ is a sequence in C such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and $Jx_n - JTx_n \to 0$ it follows that $p \in F(T)$. Here $\overset{*}{\rightharpoonup}$ denotes the weak* convergence in the dual space. A mapping $R : E \to C$ is said to be a *sunny generalized nonexpansive retraction* if the following properties are satisfied:

(1) R is generalized nonexpansive;
(2) $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \ge 0$;
(3) $Rx = x$ for all $x \in C$.

A nonempty subset C of E is said to be a *sunny generalized nonexpansive retract* (resp. *generalized nonexpansive retract*) of E if there exists a sunny generalized nonexpansive retraction (resp. generalized nonexpansive retraction) of E onto C (see [42] for more details). We know the following result.

Lemma 4.1 (*Cf. [43, Theorem 3.3]*). *Let C be a nonempty and closed subset of a reflexive, strictly convex and smooth Banach space E. Then the following are equivalent:*

(i) *C is a sunny generalized nonexpansive retract of E;*
(ii) *C is a generalized nonexpansive retract of E;*
(iii) *JC is closed and convex.*

In this case, the sunny generalized nonexpansive retraction from E onto C is given by $J^{-1}\Pi_{JC}J$, where Π_{JC} is the generalized projection from E^* onto JC.

In 2007, Ibaraki and Takahashi [44] proved that the sequence $\{x_n\}$ defined by (1.1) converges *weakly* to a fixed point of a generalized nonexpansive mapping T.

Let C be a nonempty subset of a smooth, strictly convex and reflexive Banach space E and let $T : C \to E$ be a mapping. We define the *duality* $T^* : JC \to E^*$ of T by (see [32])

$$T^*x^* = JTJ^{-1}x^* \quad \text{for all } x^* \in JC.$$

The following duality theorem is proved in [33, Theorem 20].

Lemma 4.2. *Let C be a nonempty subset of a reflexive, strictly convex and smooth Banach space E. Let $T : C \to E$ be a strongly generalized nonexpansive mapping with property* (G3) *and let $T^* : JC \to E^*$ be the duality of T. Then T^* is strongly relatively nonexpansive and $F(T^*) = JF(T)$.*

Theorem 4.3. *Let C be a nonempty, closed and sunny generalized nonexpansive retract of a uniformly smooth Banach space E whose dual space has a Fréchet differentiable norm. Let $T : C \to E$ be a strongly generalized nonexpansive mapping with property* (G3). *Let $\{x_n\}$ be a sequence in C defined by $u \in E$, $x_1 \in C$ and*

$$x_{n+1} = R_C(\alpha_n u + (1 - \alpha_n)Tx_n), \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ *satisfying Conditions* (C1) *and* (C2). *Then $\{x_n\}$ converges strongly to $R_{F(T)}u$, where $R_{F(T)}$ is the unique sunny generalized nonexpansive retraction from E onto $F(T)$.*

Proof. Suppose that $T^* : JC \to E^*$ is the duality of T. From Lemma 4.2, T^* is strongly relatively nonexpansive and $F(T^*) = JF(T)$. Let $x_n^* \equiv Jx_n$ and $u^* = Ju$. Using (4.1) and $R_C = J^{-1}\Pi_{JC}J$, we obtain

$$\begin{aligned} x_{n+1}^* &= \Pi_{JC}J(\alpha_n J^{-1}u^* + (1-\alpha_n)J^{-1}T^*x_n^*) \\ &= \Pi_{JC}J^{*-1}\left(\alpha_n J^*u^* + (1-\alpha_n)J^*T^*x_n^*\right) \end{aligned}$$

for all $n \in \mathbb{N}$. Applying Theorem 3.3 gives $x_n^* \to \Pi_{F(T^*)}u^*$. Since J^{-1} is norm-to-norm continuous,

$$x_n = J^{-1}x_n^* \to J^{-1}\Pi_{F(T^*)}u^* = J^{-1}\Pi_{JF(T)}(Ju) = R_{F(T)}u. \quad \square$$

If the mapping T in Theorem 4.3 is a self mapping, then we have the following corollary.

Corollary 4.4. *Let C be a nonempty, closed, convex and sunny generalized nonexpansive retract of a uniformly smooth Banach space E whose dual space has a Fréchet differentiable norm. Let $T : C \to C$ be a strongly generalized nonexpansive mapping with property* (G3). *Let $\{x_n\}$ be a sequence in C defined by $u, x_1 \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ *satisfying Conditions* (C1) *and* (C2). *Then $\{x_n\}$ converges strongly to $R_{F(T)}u$.*

Let C be a nonempty subset of a smooth Banach space E. Recall that a mapping $T : C \to E$ is *firmly generalized nonexpansive type* [34] if

$$\varphi(Tx, Ty) + \varphi(Ty, Tx) + \varphi(x, Tx) + \varphi(y, Ty) \le \varphi(x, Ty) + \varphi(y, Tx)$$

for all $x, y \in C$. It is not hard to show that the duality of a firmly generalized nonexpansive type mapping is firmly nonexpansive type. As a consequence of Theorem 3.9, we have the following result.

Theorem 4.5. *Let C be a nonempty, closed and sunny generalized nonexpansive retract of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to E$ be a firmly generalized nonexpansive type mapping with $F(T)$ is nonempty. Suppose that $\{\alpha_n\}$ is a sequence in* $(0, 1)$ *satisfying Conditions* (C1) *and* (C2). *Then the sequence $\{x_n\}$ defined by* (4.1) *converges strongly to $R_{F(T)}u$.*

5. Maximal monotone operators

Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a set-valued mapping with range $R(A) = \{x^* \in E^* : x^* \in Ax\}$ and domain $D(A) = \{x \in E : Ax \neq \emptyset\}$. Then the mapping A is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \ge 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* if A is monotone and there is no monotone operator from E into E^* whose graph properly contains the graph of A. It is known that if $A \subset E \times E^*$ is maximal monotone, then $A^{-1}0$ is closed and convex.

Theorem 5.1 (*Cf.* [45]). *Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a monotone operator. Then A is maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Using Theorem 5.1, we obtain that for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that

$$Jx \in Jx_r + rAx_r.$$

The single valued mapping $J_r : E \to D(A)$ by $J_r x = x_r$, that is, $J_r = (J + rA)^{-1}J$ is called the *resolvent* of A. We know that $A^{-1}0 = F(J_r)$ for all $r > 0$ (see [27,28,37] for more details).

Theorem 5.2 (*Cf. [27, Lemma 2.3]*). *Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. Let J_r be the resolvent of A, where $r > 0$. If $A^{-1}0$ is nonempty, then J_r is firmly nonexpansive type.*

Using this result and Theorem 3.9, we prove a strong convergence theorem for resolvents of maximal monotone operators in a Banach space.

Theorem 5.3. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E^*$ be a maximal monotone operator. Let J_r be the resolvent of A, where $r > 0$. Let $\{x_n\}$ be a sequence defined by $u, x_1 \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n)JJ_r x_n),$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) and (C2). If $A^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}u$.

From Theorem 5.1, if E is reflexive, strictly convex and smooth and $B \subset E^* \times E(=E^* \times E^{**})$ is a maximal monotone operator, then $R(J^{-1} + rB) = E$ for all $r > 0$. Thus, if $r > 0$ and $x \in E$, then there exists $z \in E$ such that

$$x = J^{-1}(Jx) \in J^{-1}(Jz) + rB(Jz) = z + rBJz.$$

It follows from the strict convexities of E and E^* that such a point z is unique. Thus we can define the *generalized resolvent* Q_r of B by

$$Q_r x = z = (I + rBJ)^{-1}x.$$

For more details, see [42,46].

Lemma 5.4 (*Cf. [34, Lemma 3.5]*). *Let E be a reflexive, strictly convex Banach space whose dual space has a uniformly Gâteaux differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator. Let Q_r be the generalized resolvent of B, where $r > 0$. If $B^{-1}0$ is nonempty, then Q_r is firmly generalized nonexpansive.*

Using this result and Theorem 4.5, we prove a strong convergence theorem for generalized resolvents of maximal monotone operators in a Banach space.

Theorem 5.5. *Let E be a uniformly convex and uniformly smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. Let Q_r be the generalized resolvent of B, where $r > 0$. Let $\{x_n\}$ be a sequence defined by $u, x_1 \in E$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Q_r x_n,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) and (C2). If $B^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $R_{(BJ)^{-1}0}u$, where $R_{(BJ)^{-1}0}$ is the unique sunny generalized nonexpansive retraction from E onto $(BJ)^{-1}0$.

Remark 5.6. In Theorem 5.5, we present a strong convergence theorem for the generalized resolvent with a new control condition. This is complementary to Ibaraki and Takahashi's result [46, Theorem 4.2].

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References

[1] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.
[2] A. Genel, J. Lindenstrauss, An example concerning fixed points, Israel J. Math. 22 (1975) 81–86.
[3] B. Halpren, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc. 73 (1967) 957–961.
[4] H.H. Bauschke, P.L. Combettes, A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, Math. Oper. Res. 26 (2001) 248–264.
[5] S. Matsushita, W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in a Banach space, Fixed Point Theory Appl. 2004 (2004) 37–47.
[6] S. Matsushita, W. Takahashi, An iterative algorithm for relatively nonexpansive mappings by hybrid method and applications, in: Proceedings of the Third International Conference on Nonlinear Analysis and Convex Analysis, 2004, pp. 305–313.
[7] S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, J. Approx. Theory 134 (2005) 257–266.

[8] Y.I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in: Lecture Notes in Pure and Appl. Math., vol. 178, Dekker, New York, 1996, pp. 15–50.
[9] S. Reich, A weak convergence theorem for the alternating method with Bregman distances, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in: Lecture Notes in Pure and Appl. Math., vol. 178, Dekker, New York, 1996, pp. 313–318.
[10] K. Aoyama, F. Kohsaka, W. Takahashi, Strongly nonexpansive sequences in Banach spaces and applications, J. Fixed Point Theory Appl. 5 (2009) 201–225.
[11] Y. Kimura, W. Takahashi, On a hybrid method for a family of relatively nonexpansive mappings in a Banach space, J. Math. Anal. Appl. 357 (2) (2009) 356–363.
[12] F. Kohsaka, W. Takahashi, Approximating common fixed points of countable families of strongly nonexpansive mappings, Nonlinear Stud. 14 (2007) 219–234.
[13] G. Lewicki, G. Marino, On some algorithms in Banach spaces finding fixed points of nonlinear mappings, Nonlinear Anal. 71 (2009) 3964–3972.
[14] X. Li, N. Huang, D. O'Regan, Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications, Comput. Math. Appl. 60 (2010) 1322–1331.
[15] W. Nilsrakoo, S. Saejung, Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings, Fixed Point Theory Appl. (2008) Art. ID 312454, 19 p.
[16] W. Nilsrakoo, S. Saejung, Strong convergence theorems by Halpern–Mann iterations for relatively nonexpansive mappings in Banach spaces, Appl. Math. Comput. 217 (2011) 6577–6586.
[17] X. Qin, Y.J. Cho, S.M. Kang, H. Zhou, Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings, Appl. Math. Lett. 22 (2009) 1051–1055.
[18] X. Qin, Y. Su, Strong convergence theorems for relatively nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007) 1958–1965.
[19] S. Reich, S. Sabach, Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces, Nonlinear Anal. 73 (2010) 122–135.
[20] Y. Su, D. Wang, M. Shang, Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings, Fixed Point Theory Appl. (2008) Art. ID 284613, 8 p.
[21] C. Zhang, J. Li, B. Liu, Strong convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Comput. Math. Appl. 61 (2011) 262–276.
[22] Y. Censor, S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, Optimization 37 (1996) 323–339.
[23] R.E. Bruck, S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, Houston J. Math. 3 (1977) 459–470.
[24] S. Reich, A limit theorem for projections, Linear Multilinear Algebra 13 (1983) 281–290.
[25] K. Aoyama, W. Takahashi, Strong convergence theorems for a family of relatively nonexpansive mappings in Banach spaces, Fixed Point Theory 8 (2007) 143–160.
[26] M.B. Lee, S.H. Park, Convergence of sequential parafirmly nonexpansive mappings in reflexive Banach spaces, J. Optim. Theory Appl. 123 (2004) 549–571.
[27] F. Kohsaka, W. Takahashi, Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces, SIAM J. Optim. 19 (2008) 824–835.
[28] F. Kohsaka, W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, Arch. Math. (Basel) 91 (2008) 166–177.
[29] S. Reich, S. Sabach, Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces, in: Fixed-Point Algorithms for Inverse Problems in Science and Engineering, vol. 49, Springer, New York, 2011.
[30] S. Reich, S. Sabach, A projection method for solving nonlinear problems in reflexive Banach spaces, J. Fixed Point Theory Appl. 9 (2011) 101–116.
[31] H.H. Bauschke, J.M. Borwein, P.L. Combettes, Bregman monotone optimization algorithms, SIAM J. Control Optim. 42 (2003) 596–636.
[32] T. Honda, T. Ibaraki, W. Takahashi, Duality theorems and convergence theorems for nonlinear mappings in Banach spaces and applications, Int. J. Math. Stat. 6 (2010) 46–64.
[33] W. Nilsrakoo, S. Saejung, On the fixed-point set of a family of relatively nonexpansive and generalized nonexpansive mappings, Fixed Point Theory Appl. (2010) Art. ID 414232, 14 p.
[34] T. Ibaraki, W. Takahashi, Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces, J. Nonlinear Convex Anal. 10 (2009) 21–32.
[35] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
[36] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002) 938–945.
[37] F. Kohsaka, W. Takahashi, Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, Abstr. Appl. Anal. 2004 (2004) 239–249.
[38] H.K. Xu, Another control condition in an iterative method for nonexpansive mappings, Bull. Aust. Math. Soc. 65 (2002) 109–113.
[39] P.E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (2008) 899–912.
[40] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301–308.
[41] S. Saejung, Halpern's iteration in Banach spaces, Nonlinear Anal. 73 (2010) 3431–3439.
[42] T. Ibaraki, W. Takahashi, A new projection and convergence theorems for projections in Banach spaces, J. Approx. Theory 149 (2007) 1–14.
[43] F. Kohsaka, W. Takahashi, Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces, J. Nonlinear Convex Anal. 8 (2007) 197–209.
[44] T. Ibaraki, W. Takahashi, Weak convergence theorem for new nonexpansive mappings in Banach spaces and its applications, Taiwanese J. Math. 11 (2007) 929–944.
[45] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970) 75–88.
[46] T. Ibaraki, W. Takahashi, Weak and strong convergence theorems for new resolvents of maximal monotone operators in Banach spaces, Adv. Math. Econ. 10 (2007) 51–64.